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AXISYMMETRIC CONTACT PROBLEMS FOR A TWO-LAYER ELASTIC HALF-SPACE WITH AN ANNULAR OR CIRCULAR CRACK AT THE INTERFACE OF THE LAYERS†

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Exact analytical solutions of three related axisymmetric mixed problems of the theory of elasticity are given. These problems are concerned with the pressure of an annular or circular punch on a two-layer half-space taking into account cohesion and the existence of an annular or circular crack at the interface of the layers. The formulation and solution of these problems are based on a special regularization of the solution of the first fundamental boundary-value problem of the theory of elasticity for a single layer when there are arbitrary normal and shear loads on its boundary planes. The solution is constructed by the Hankel transformation method which ensures that the integrals converge for all stresses and displacements. © 2002 Elsevier Science Ltd. All rights reserved.

The method of regularization of the basic solution, proposed in [1] and developed in [2] enables one to reduce the initial problems to well-studied‡ resolving systems of singular integral equations. This approach is extended below to the formulation and solution of more complex contact problems for a multilayer half-space with cracks at the boundaries of the layers.

1. THE FORMULATION OF MIXED PROBLEMS

The two-layer space under consideration consists of a layer of arbitrary thickness H and a base layer of infinite thickness (a uniform half-space) which are given the numbers 1 and 2, respectively. The moduli of elasticity E_i and Poisson's ratios $\nu_i (i = 1, 2)$ of the layers can take different and arbitrary values. We take the origin of a cylindrical system of coordinates r, z as being in the interface plane of the layers with the OZ axis directed upwards and orthogonal to the layers. In this system of coordinates, the upper layer $0 \leqslant z \leqslant H$ and the base layer $z \leqslant 0$ are separated by the plane $z = 0$, and the plane $z = H$ is the upper boundary of the upper layer (Fig. 1).

The pressure P of an annular or circular coupled punch $r_1 \leqslant r \leqslant r_2 (r_1 \geqslant 0)$ is applied to the boundary $z = H$, there is an annular or circular crack $r_3 \leqslant r \leqslant r_4 (r_3 \geqslant 0)$ in the interface plane $z = 0$ and, outside the crack, the conditions for rigid adhesion of the layers, which ensure the continuity of the normal and shear stresses and displacements, must be satisfied. In the first problem, the punch and the crack the annular, where $r_3 > r_2$ (version 1) and $r_4 < r_1$ (version 2). In the second problem, the punch is annular and the crack is circular $r_3 = 0, r_4 < r_1$ and, in the third problem, the punch is circular ($r_1 = 0$) and the crack is annular ($r_3 > r_2$). In all of the above-mentioned problems, arbitrarily specified symmetric normal and shear loads are applied to the edges of the cracks. We are particularly interested in problems with load-free cracks which, under the action of a punch, are capable of opening in the case of specific elastic and geometric characteristics. These characteristics can only be established by numerical solution of the problems, thereby confirming the correctness of their formulation, as has been shown previously [1].

The regularized basic solution of the fundamental boundary-value problem for a single layer described below and the solutions of mixed problems for a two-layer half-space are given in the dimensionless variables $\rho = r/a$, $t = z/H$, where a is a certain value of the radius r, which is taken as the linear unit of measurement. In the numerical solutions of the problems, it is convenient to equate the quantity a, for

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‡NIKISHIN, V. S., Correct formulation and numerical solution of fundamental and mixed problems of the theory of elasticity for multilayer and continuously inhomogeneous media. Doctorate dissertation, 01.01.07. Computing Centre, Academy of Sciences of the USSR, 1982.

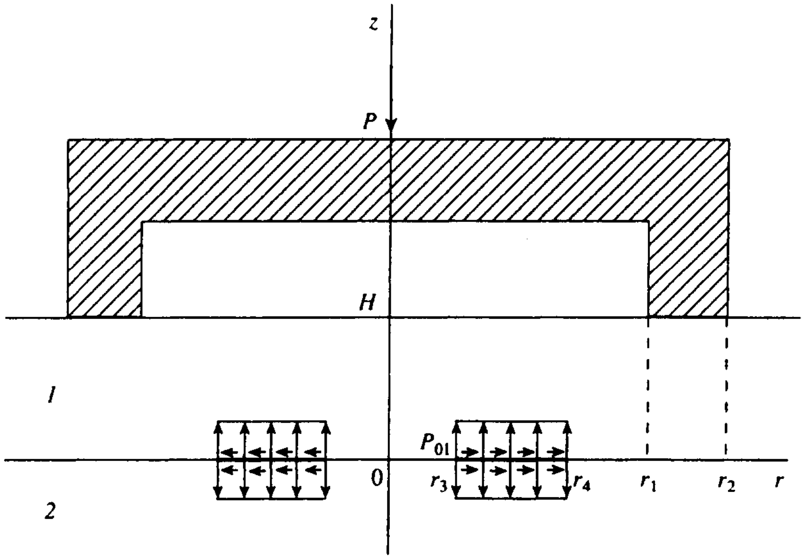

Fig. 1

example, to r_2 or r_4 (Fig. 1). The magnitudes of the ratios $\lambda = H/a$, $\rho_j = r_j/a$ ($j = 1, 2, 3, 4$), $\delta = E_1/E_2$, $\chi = \delta(1 + \nu_2)/(1 + \nu_1)$ are the characteristic geometrical and elastic parameters of mixed problems. On the dimensionless axis Ot, the upper layer ($i = 1$) is located in the interval $0 \leqslant t \leqslant 1$ and the base layer ($i = 2$) is located in the unbounded interval $-\infty < t \leqslant 0$. We will denote the normal and shear stresses and the axial and radial displacements in a layer by $\sigma_{zi}(\rho, t)$, $\tau_{rzi}(\rho, t)$, $w_i(\rho, t)$ and $u_i(\rho, t)$, where $i = 1, 2$ is the number of a layer.

2. THE REGULARIZED BASIC SOLUTION OF MIXED PROBLEMS

The regularized basic solution of mixed problems for a two-layer half-space was constructed [2] with the following boundary conditions on the external surface $t = 1$ and on the interface of the layers $t = 0$

$$\sigma_{z1} = p_1(\rho) + p_1^*(\rho), \quad \tau_{rz1} = q_1(\rho) + q_1^*(\rho) \quad \text{when } t = 1 \tag{2.1}$$

$$\sigma_{z1} = \sigma_{z2} = p_0(\rho), \quad \tau_{rz1} = \tau_{rz2} = q_0(\rho) \quad \text{when } t = 0 \tag{2.2}$$

where $\rho_j(\rho)$ and $q_j(\rho)$ ($j = 0, 1$) are arbitrary functions on the semi axis $0 \leqslant \rho < \infty$, which can be represented by the Hankel integrals

$$p_j(\rho) = \int_0^\infty \beta \bar{p}_j(\beta) J_0(\rho\beta) d\beta, \quad q_j(\rho) = \int_0^\infty \beta \bar{q}_j(\beta) J_1(\rho\beta) d\beta \tag{2.3}$$

$$\bar{p}_j(\beta) = \int_0^\infty \rho p_j(\rho) J_0(\rho\beta) d\rho, \quad \bar{q}_j(\beta) = \int_0^\infty \rho q_j(\rho) J_1(\rho\beta) d\rho \tag{2.4}$$

$p_1^*(\rho)$ and $q_1^*(\rho)$ ($0 \leqslant \rho < \infty$) are functions, which are as small as desired, intended for the regularization of the solution of problem (2.1)–(2.4) in the case of a single upper layer ($i = 1$). They will be presented and justified below. The following constructive representation of the axial and radial displacements $w_i(\rho, t)$ and $u_i(\rho, t)$ ($i = 1, 2$) on the boundaries of the layers $t = 1$ and $t = 2$ are required from this solution for the mathematical formulation of the mixed problems considered

$$w_i(\rho, t) = \int_0^\infty \Delta_{wi}(t, \beta) J_0(\rho\beta) d\beta, \quad u_i(\rho, t) = \int_0^\infty \Delta_{ui}(t, \beta) J_1(\rho\beta) d\beta \tag{2.5}$$

$$\Delta_{vi}(t,\beta)=E_i^{-1}(1+\nu_i)aD_{vi}(t,\beta),\quad i=1,2$$
$$D_{v1}(k,\beta)=\Delta_{vpk1}(\beta)\bar{p}_1(\beta)+\Delta_{vpk0}(\beta)\bar{p}_0(\beta)+\Delta_{vqk1}(\beta)\bar{q}_1(\beta)+\Delta_{vqko}(\beta)\bar{q}_0(\beta),$$
$$v=w,u;\ k=0,1 \tag{2.6}$$
$$D_{v2}(0,\beta)=\tilde{\Delta}_{vp00}\bar{p}_0(\beta)+\tilde{\Delta}_{vq00}\bar{q}_0(\beta),\quad v=w,u$$

The functions $\Delta_{\upsilon skm}(\beta)$ and the constants $\Delta_{\upsilon s00}$ ($\upsilon=w,u;s=p,q;k,m=0,1$) are given by the following formulae

$$\Delta_p\Delta_{wp11}=a_1(\xi_3\xi_4+4\xi_1\xi_2^2)$$
$$\Delta_p\Delta_{wp01}=-\Delta_p\Delta_{wp10}=2a_1\xi_2(\xi_3+\xi_1\xi_4)$$
$$\Delta_q\Delta_{wq11}=\Delta_p\Delta_{up11}=b_1\xi_3^2+4\xi_1^2\xi_2^2$$
$$\Delta_q\Delta_{wq10}=\Delta_q\Delta_{wq01}=-\Delta_p\Delta_{up01}=2a_1\xi_1\xi_2\xi_3$$
$$\Delta_p\Delta_{up10}=2a_1\xi_2(R(b_1\xi_3-\xi_1\xi_4)-\xi_1\xi_3)$$
$$\Delta_q\Delta_{uq11}=a_1(\xi_3\xi_4-4\xi_1\xi_2^2)$$
$$\Delta_q\Delta_{uq10}=-2a_1\xi_2((b_1R+1)\xi_3+(R-1)\xi_1\xi_4)$$
$$\Delta_p\Delta_{wp00}=-a_1(\xi_3(\xi_4-R\xi_3)+4\xi_1\xi_2^2)$$
$$\Delta_q\Delta_{wq00}=b_1(1+b_1R)\xi_3^2+4(1-R)\xi_1^2\xi_2^2 \tag{2.7}$$
$$\Delta_p\Delta_{up00}=b_1\xi_3(\xi_3+a_1R\xi_4)+4\xi_1\xi_2^2(\xi_1-a_1R)$$
$$\Delta_q\Delta_{uq01}=2a_1\xi_2(\xi_3-\xi_1\xi_4)$$
$$\Delta_q\Delta_{uq00}=a_1(4(1-R)\xi_1\xi_2^2-(1+b_1R)\xi_3\xi_4)$$
$$\Delta_p=\xi_3(\xi_3+a_1R\xi_4)+4\xi_1\xi_2^2(a_1R-\xi_1)$$
$$\Delta_q=(1+b_1R)\xi_3^2+4(R-1)\xi_1^2\xi_2^2$$
$$\tilde{\Delta}_{wp00}=\tilde{\Delta}_{uq00}=2(1-\nu_2),\quad \tilde{\Delta}_{up00}=\tilde{\Delta}_{wq00}=1-2\nu_2$$
$$a_1=2(1-\nu_1),\quad b_1=1-2\nu_1,\quad \xi_1=\lambda\beta,\quad \xi_2=\exp(-\lambda\beta),\quad \xi_3=1-\xi_2^2,\quad \xi_4=1+\xi_2^2$$

The representation of the intensity functions of the normal and shear loads in boundary conditions (2.1) on the semi-axis $0\leqslant\rho<\infty$, corresponding to solution (2.5)–(2.7) have the form

$$p_1^*(\rho)=-\int_0^\infty \beta R(\beta)\Delta_{wp11}(\beta)J_0(\rho\beta)d\beta$$
$$q_1^*(\rho)=-\int_0^\infty \beta R(\beta)\Delta_{wq11}(\beta)J_1(\rho\beta)d\beta \tag{2.8}$$

where $\Delta_{ws11}(\beta)$ ($s=p,q$) are functions which are represented by formulae from (2.7), where it is assumed that

$$R(\beta)=\varepsilon\exp(-n\beta),\quad 0<\varepsilon\ll1,\quad n\gg1 \tag{2.9}$$

The introduction of the load intensity functions $p_1^*(\rho)$ and $q_1^*(\rho)$ (2.8) into boundary conditions (2.1) is solely intended to ensure, through the function $R(\beta)$, the convergence of the integrals (2.5) which, in the case when $R(\beta)\equiv0$ and, therefore, when $p_1^*(\rho)\equiv q_1^*(\rho)\equiv0$, diverge at the lower limits. However, it is required here that the moduli $|p_1^*(\rho)|$ and $|q_1^*(\rho)|$ ($0\leqslant\rho<\infty$) and the modulus of the principal overload vector $p_1^*(\rho)$ (the principal overload vector $q_1^*(\rho)$ vanishes by the symmetry condition) should not exceed the quantity $\delta=\delta(\varepsilon,n)>0$, which is as small as desired and depends on the constants ε

and n of the function $R(\beta)$ (2.9). When $\varepsilon \to 0$ and $n \to \infty$, we have $\delta(\varepsilon, n) = O(n^{-1}\sqrt{\varepsilon/n})$, and, consequently, the functions $p_1^*(\rho)$, $q_1^*(\rho)$ in boundary conditions (2.1) can be interpreted as infinitesimal functions of the regularization of the basic solution while retaining its form and an accuracy which may be as high as desired.

To formulate the mixed problems, we write out the principal and higher terms of the asymptotic expansions of the functions (2.7) as $\beta \to \infty$

$$\begin{aligned}
&\Delta_{wp11},\ \Delta_{uq11},\ -\Delta_{wp00},\ -\Delta_{uq00} \sim 2(1-\nu_1)+8(1-\nu_1)\xi_1^2(\beta)\xi_2^2(\beta)\\
&\Delta_{wq11},\ \Delta_{up11},\ \Delta_{wq00},\ \Delta_{up00} \sim 1-2\nu_1+8(1-\nu_1)\xi_1^2(\beta)\xi_2^2(\beta)\\
&-\Delta_{wp10},\ \Delta_{wq10},\ -\Delta_{up10},\ \Delta_{uq10}\\
&\Delta_{wp01},\ \Delta_{wq01},\ -\Delta_{up01},\ -\Delta_{uq01} \sim 4(1-\nu_1)\xi_1(\beta)\xi_2(\beta)
\end{aligned} \tag{2.10}$$

3. FORMULATION AND SOLUTION OF THE MIXED PROBLEM FOR AN ANNULAR PUNCH AND CRACK

Non-zero axial displacements and zero radial displacements are specified in the external boundary of the two-layer half-space $t = 1$ in the contact area for the coupled punch $L_1 = (0 < \rho_1 \leqslant \rho \leqslant \rho_2)$

$$\frac{E_1}{(1+\nu_1)a}w_1 = -h+\gamma(\rho), \quad u_1 \equiv 0 \tag{3.1}$$

and zero axial and shear stresses

$$\sigma_{z1} = 0, \quad \tau_{rz1} = 0, \quad \rho \in L_2 \tag{3.2}$$

outside the contact area for the punch $L_2 = (0 \leqslant \rho < \rho_1, \rho_2 < \rho < \infty)$.

We will now explain that, under conditions (3.1), the constant h is expressed in terms of the depth of loading of the punch into the layer w_1° according to the formula $h = E_1w_1^\circ/[a(1 + \nu_1)] > 0$, while the function $\gamma(\rho) \geqslant 0$ describes the surface of the base (the bottom) of the punch.

The axial and shear stresses

$$\sigma_{z1} = \sigma_{z2} = p_{01}(\rho), \quad \tau_{rz1} = \tau_{rz2} = q_{01}(\rho), \quad \rho \in L_3 \tag{3.3}$$

are specified on the interface of the layers $t = 0$ in the upper and lower edges of the annular crack in the region $L_3 = (0 < \rho_3 \leqslant \rho \leqslant \rho_4)$ and the conditions

$$w_1 = w_2, \quad u_1 = u_2, \quad \rho \in L_4 \tag{3.4}$$

for the continuity of the axial and radial displacements must be satisfied outside the crack in the region $L_4 = (0 \leqslant \rho < \rho_3, \rho_4 < \rho < \infty)$.

Furthermore, in the region L_4, the basic solution automatically satisfies the conditions for the continuity of the axial and shear stresses

$$\sigma_{z1} = \sigma_{z2}, \quad \tau_{rz1} = \tau_{rz2}, \quad \rho \in L_4 \tag{3.5}$$

Substituting the formulae of the basic solution (2.1)–(2.6) into boundary conditions (3.1)–(3.4), we obtain the following system of dual integral equations in terms of the Hankel transform

$$\int_0^\infty D_{w1}(1,\beta)J_0(\rho\beta)d\beta = -h+\gamma(\rho), \quad \int_0^\infty D_{u1}(1,\beta)J_1(\rho\beta)d\beta = 0, \quad \rho \in L_1 \tag{3.6}$$

$$\int_0^\infty \beta\bar{p}_1(\beta)J_0(\rho\beta)d\beta = 0, \quad \int_0^\infty \beta\bar{q}_1(\beta)J_1(\rho\beta)d\beta = 0, \quad \rho \in L_2 \tag{3.7}$$

$$\int_0^\infty \beta\bar{p}_0(\beta)J_0(\rho\beta)d\beta = p_{01}(\rho), \quad \int_0^\infty \beta\bar{q}_0(\beta)J_1(\rho\beta)d\beta = q_{01}(\rho), \quad \rho \in L_3 \tag{3.8}$$

$$\int_0^\infty D_{w12}(0,\beta)J_0(\rho\beta)d\beta = 0, \quad \int_0^\infty D_{u12}(0,\beta)J_1(\rho\beta)d\beta = 0, \quad \rho \in L_4 \tag{3.9}$$

The functions $D_{\upsilon 12}(0, \beta) = D_{\upsilon 1}(0, \beta) - \chi D_{\upsilon 2}(0, \beta)$ ($\upsilon = w, u$) are represented by formula (2.6) for $D_{\upsilon 1}(0, \beta)$, taking into account the replacement of $\Delta_{\upsilon s00}(\beta)$ ($\upsilon = w, u; s = p, q$) by the following functions $\Delta_{\upsilon s0}(\beta)$

$$\begin{aligned} &\Delta_{wp0} = \Delta_{wp00} - 2(1-\nu_2)\chi, \quad \Delta_{wq0} = \Delta_{wq00} - (1-2\nu_2)\chi \\ &\Delta_{up0} = \Delta_{op00} - (1-2\nu_2)\chi, \quad \Delta_{uq0} = \Delta_{uq00} - 2(1-\nu_2)\chi \end{aligned} \tag{3.10}$$

Taking boundary conditions (3.2) into account, we write their Hankel transforms in the form

$$\bar{p}_1(\beta) = \int_{L_1} \rho p_1(\rho)J_0(\rho\beta)d\rho, \quad \bar{q}_1(\beta) = \int_{L_1} \rho q_1(\rho)J_1(\rho\beta)d\rho \tag{3.11}$$

It can be seen that, according to the inversion theorem Hankel transforms (3.11) automatically satisfy conditions (3.7) and, therefore, these conditions are excluded from any further consideration of the system of dual integral equations (3.6)–(3.9).

When account is taken of (3.11), the remaining system of dual integral equations (3.6), (3.8) and (3.9) can be immediately reduced to a system of singular integral equations on the unbounded contour (L_1, L_4), which give rise to additional difficulties in the analytical and numerical investigation of the system of singular integral equations. It is therefore advisable initially to transform the system of dual integral equations (3.6), (3.8) and (3.9) to an equivalent system of dual integral equations, that subsequently reduces to a system of singular integral equations on the bounded contour (L_1, L_3) which is convenient for analytical and numerical investigations. In order to implement this approach, in the system of dual integral equations (3.6), (3.8) and (3.9) we will change from the transforms $\bar{p}_0(\beta)$ and $\bar{q}_0(\beta)$ to the new Hankel transforms $\bar{f}(\beta)$ and $\bar{g}(\beta)$ using the formulae

$$\bar{f}(\beta) = D_{u12}(0,\beta), \quad \bar{g}(\beta) = D_{w12}(0,\beta) \tag{3.12}$$

The Hankel transforms $\bar{f}(\beta), \bar{g}(\beta)$ correspond to specific unknown functions $f(\rho), g(\rho)$ on the contour L_3 and are represented by the Hankel integrals

$$\bar{f}(\beta) = \int_{L_3} \rho f(\rho)J_0(\rho\beta)d\rho, \quad \bar{g}(\beta) = \int_{L_3} \rho g(\rho)J_1(\rho\beta)d\rho \tag{3.13}$$

When account is taken of the above-mentioned constructive expressions for the functions $D_{\upsilon 12}(0, \beta)$ ($\upsilon = w, u$), equalities (3.12) represent a system of functional equations in the transforms $\bar{f}(\beta), \bar{g}(\beta), \bar{p}_k(\beta), \bar{q}_k(\beta)$ ($k = 0, 1$).

From this system, we find expressions for the transforms $\bar{p}_0(\beta)$ and $\bar{q}_0(\beta)$ in terms of $\bar{f}(\beta), \bar{g}(\beta), \bar{p}_1(\beta)$ and $\bar{q}_1(\beta)$

$$\bar{p}_0(\beta) = N_u(\beta), \quad \bar{q}_0(\beta) = N_w(\beta) \tag{3.14}$$

$$N_\upsilon(\beta) = \Delta_{\upsilon f}\bar{f}(\beta) + \Delta_{\upsilon g}\bar{g}(\beta) - B_{\upsilon p}\bar{p}_1(\beta) - B_{\upsilon q}\bar{q}_1(\beta) \quad (\upsilon = w, u) \tag{3.15}$$

where

$$\begin{aligned} &\Delta_{uf} = \Delta_{wq0} / \Delta_{pq}, \quad \Delta_{ug} = -\Delta_{uq0} / \Delta_{pq} \\ &\Delta_{wf} = -\Delta_{wp0} / \Delta_{pq}, \quad \Delta_{wg} = \Delta_{up0} / \Delta_{pq} \\ &\Delta_{pq} = \Delta_{up0}\Delta_{wq0} - \Delta_{uq0}\Delta_{wp0} \end{aligned} \tag{3.16}$$

$$B_{\upsilon s} = \Delta_{\upsilon f}\Delta_{us01} + \Delta_{\upsilon g}\Delta_{ws01}, \quad \upsilon = w, u; \ s = p, q \tag{3.17}$$

Substituting formulae (3.14) into Eqs (3.6) and (3.8) and formulae (3.12) into Eq. (3.9), we obtain the following system of dual integral equations in the transforms $\bar{p}_1(\beta), \bar{q}_1(\beta), \bar{f}(\beta), \bar{g}(\beta)$

$$\int_0^\infty M_w(\beta)J_0(\rho\beta)d\beta = -h+\gamma(\rho), \quad \int_0^\infty M_u(\beta)J_1(\rho\beta)d\beta = 0, \quad \rho \in L_1 \tag{3.18}$$

$$\int_0^\infty \beta N_u(\beta)J_0(\rho\beta)d\beta = p_{01}(\rho), \quad \int_0^\infty \beta N_w(\beta)J_1(\rho\beta)d\beta = q_{01}(\rho), \quad \rho \in L_3 \tag{3.19}$$

$$\int_0^\infty \bar{g}(\beta)J_0(\rho\beta)d\beta = 0, \quad \int_0^\infty \bar{f}(\beta)J_1(\rho\beta)d\beta = 0, \quad \rho \in L_4 \tag{3.20}$$

where

$$M_v(\beta) = \Delta_{vp}\bar{p}_1(\beta)+\Delta_{vq}\bar{q}_1(\beta)+B_{vf}\bar{f}(\beta)+B_{vg}\bar{g}(\beta), \quad v = w, u \tag{3.21}$$

$$\Delta_{vs} = \Delta_{vs11} - \Delta_{vp10}B_{us} - \Delta_{vq10}B_{ws}, \quad v = w, u; \quad s = p, q \tag{3.22}$$

$$B_{vl} = \Delta_{vp10}\Delta_{ul} + \Delta_{vq10}\Delta_{wl}, \quad v = w, u; \quad l = f, g \tag{3.23}$$

We then carry out the following transformations of Eq. (3.19) and (3.20): we multiply the first equation of (3.19) by ρ and integrate with respect to ρ within the limits from ρ_3 to ρ and then divide by ρ; we integrate the second equation of (3.19) with respect to ρ within the limits from ρ_3 to ρ, differentiate the first equation of (3.20) with respect to ρ, multiply the second equation of (3.20) by ρ and differentiate with respect to ρ and then divide it by ρ. As a result, the transformed equations have the form

$$\int_0^\infty N_u(\beta)J_1(\rho\beta)d\beta = G(\rho)+\frac{D}{\rho}, \quad \int_0^\infty N_w(\beta)J_0(\rho\beta)d\beta = F(\rho)+C, \quad \rho \in L_3 \tag{3.24}$$

$$\int_0^\infty \beta\bar{g}(\beta)J_1(\rho\beta)d\beta = 0, \quad \int_0^\infty \beta\bar{f}(\beta)J_0(\rho\beta)d\beta = 0, \quad \rho \in L_4 \tag{3.25}$$

$$G(\rho) = \frac{1}{\rho}\int_{\rho_3}^{\rho} xp_{01}(x)dx, \quad F(\rho) = -\int_{\rho_3}^{\rho} q_{01}(x)dx \tag{3.26}$$

D and C are arbitrary constants which are determined below. According to the inversion theorem Hankel transforms $\bar{g}(\beta)$ and $\bar{f}(\beta)$ given by (3.13) automatically satisfy Eqs (3.25) and, consequently, the unknown transforms $\bar{p}_1(\beta), \bar{q}_1(\beta), \bar{f}(\beta)$ and $\bar{g}(\beta)$ are now determined by the system of equations (3.18) and (3.24) only on the bounded contours $L_1 = (0 < \rho_1 \leqslant \rho \leqslant \rho_2)$ and $L_3 = (0 < \rho_3 \leqslant \rho \leqslant \rho_4)$.

Next, using the asymptotic form (2.10) and equalities (3.10), we separate out the principal terms of the functions $\Delta_{vr}(\beta)$ ($v = w, u; r = p, q, f, g$) (3.22), (3.16) when $\beta \to \infty$

$$\begin{aligned} (\Delta_{wp}, \Delta_{uq}) &= a_1 + (\Delta^*_{wp}, \Delta^*_{uq}) \quad (\Delta_{wq}, \Delta_{up}) = b_1 + (\Delta^*_{wq}, \Delta^*_{up}) \\ (\Delta_{wf}, \Delta_{ug}) &= a_2 + (\Delta^*_{wf}, \Delta^*_{ug}) \quad (\Delta_{wg}, \Delta_{uf}) = b_2 + (\Delta^*_{wg}, \Delta^*_{uf}) \end{aligned} \tag{3.27}$$

where the constants a_1 and b_1 are determined by formulae (2.7) and

$$\begin{aligned} &a_2 = c/(d^2-c^2), \quad b_2 = d/(d^2-c^2) \\ &c = 2(1-\nu_1+(1-\nu_2)\chi), \quad d = 1-2\nu_1-(1-2\nu_2)\chi \end{aligned} \tag{3.28}$$

When $\beta \to \infty$, the functions $\Delta^*_{vr}(\beta)$ and $B_{vr}(\beta)$ ($v = w, u; r = p, q, f, g$), defined by formulae (3.27), (3.23) and (3.17), are of the order of infinitesimal functions

$$\Delta^*_{vr} = O(\xi_1^2(\beta)\xi_2^2(\beta)), \quad B_{vr} = O(\xi_1(\beta)\xi_2(\beta)) \tag{3.29}$$

The mathematical apparatus used to investigate the analogous system of equations in the basic mixed problem can be extended completely to system of integral equations (3.18), (3.24), taking account of formulae (3.27). On applying it, we reduce the above-mentioned system of equations for the transforms to the following system of two integral equations, with Cauchy kernels for complex functions of the real variable $w_j(x) = \varphi_j(x) + i\vartheta_j(x)$ and their conjugates $\bar{\omega}_j(x) = \bar{\varphi}_j(x) - i\bar{\vartheta}_j(x)$ ($j = 1, 2$), which solves the initial problem,

$$a_j\omega_j(x)+\frac{b_j}{\pi i}\int\limits_{\rho_{2j-1}}^{\rho_{2j}}\frac{\omega_j(t)}{t-x}dt+\frac{1}{\pi}\sum_{n=1}^{2}\int\limits_{\rho_{2n-1}}^{\rho_{2n}}\frac{H_{jn}^{+}(x,t)\omega_n(t)+H_{jn}^{-}(x,t)\overline{\omega_n(t)}}{\sqrt{(x-\rho_{2j-1})(t-\rho_{2n-1})}}dt=$$

$$=\frac{2}{\pi}\frac{\Phi_j(x)}{\sqrt{x-\rho_{2j-1}}} \tag{3.30}$$

Here

$$H_{jn}^{\pm}(x,t)=K_{00jn}(x,t)\pm K_{11jn}(x,t)+i[K_{10jn}(x,t)\mp K_{01jn}(x,t)] \tag{3.31}$$

$$K_{00jn}=xtG_{00jn}+a_jM_{00jn},\quad K_{01jn}=xG_{01jn}+b_jM_{01jn}$$
$$K_{10jn}=tG_{10jn}+b_jM_{10jn},\quad K_{11jn}=G_{11jn}+a_jM_{11jn} \tag{3.32}$$

$$G_{kmjn}=\int\limits_0^\infty b_{kmjn}S_{kj}(x,\beta)S_{mn}(t,\beta)d\beta,\quad k,m=0,1$$
$$b_{0011}=\Delta_{wp}^*(\beta),\quad b_{0111}=\Delta_{wq}^*(\beta),\quad b_{1011}=\Delta_{up}^*(\beta),\quad b_{1111}=\Delta_{uq}^*(\beta)$$
$$b_{0012}=B_{wf}(\beta),\quad b_{0112}=B_{wg}(\beta),\quad b_{1012}=B_{uf}(\beta),\quad b_{1112}=B_{ug}(\beta) \tag{3.33}$$
$$b_{0021}=-B_{wp}(\beta),\quad b_{0121}=-B_{wq}(\beta),\quad b_{1021}=-B_{up}(\beta),\quad b_{1121}=-B_{uq}(\beta)$$
$$b_{0022}=\Delta_{wf}^*(\beta),\quad b_{0122}=\Delta_{wg}^*(\beta),\quad b_{1022}=\Delta_{uf}^*(\beta),\quad b_{1122}=\Delta_{ug}^*(\beta)$$

$$S_{0j}(x,\beta)=\frac{J_0(\rho_{2j-1}\beta)}{\sqrt{x+\rho_{2j-1}}}-\beta\sqrt{x-\rho_{2j-1}}\int\limits_{\rho_{2j-1}}^{x}\frac{J_1(\rho\beta)}{\sqrt{x^2-\rho^2}}d\rho$$

$$S_{1j}(x,\beta)=\frac{\rho_{2j-1}J_1(\rho_{2j-1}\beta)}{\sqrt{x+\rho_{2j-1}}}+\beta\sqrt{x-\rho_{2j-1}}\int\limits_{\rho_{2j-1}}^{x}\frac{\rho J_0(\rho\beta)}{\sqrt{x^2-\rho^2}}d\rho \tag{3.34}$$

$$M_{kkjj}=\frac{2m_{kkjj}(x,t)}{\pi\eta_1^{+}(x,t)},\quad k=0,1$$

$$M_{01jj}=\frac{\eta_1^{-}(x,t)}{t-x}\eta_2(x,t),\quad M_{10jj}=-\frac{\eta_1^{-}(x,t)}{t-x}\eta_2(t,x)$$

$$m_{00jj}=\frac{xt}{2(x^2-t^2)}\left(\frac{1}{x}\eta_3(x)-\frac{1}{t}\eta_3(t)\right) \tag{3.35}$$

$$m_{11jj}=-\rho_{2j-1}+\frac{1}{2(x^2-t^2)}(x\eta_3(x)-t\eta_3(t))$$

$$\eta_1^{\pm}(x,t)=\sqrt{(x\pm\rho_{2j-1})(t\pm\rho_{2j-1})},\quad \eta_2(x,t)=\frac{x}{t+x}\sqrt{\frac{t^2-\rho_{2j-1}^2}{x^2-\rho_{2j-1}^2}}-\frac{1}{2}$$

$$\eta_3(x)=(x^2-\rho_{2j-1}^2)\ln\frac{x+\rho_{2j-1}}{x-\rho_{2j-1}}$$

$$M_{kljn} \equiv 0\ j \neq n; \quad k, l = 0, 1$$

$$\Phi_1(x) = (\gamma(\rho_1) - h)\frac{x}{\sqrt{x+\rho_1}} + x\sqrt{x-\rho_1}\int_{\rho_1}^{x} \frac{\gamma'(\rho)d\rho}{\sqrt{x^2-\rho^2}} \tag{3.36}$$

$$\Phi_2(x) = \sqrt{x-\rho_3}\left(-x\int_{\rho_3}^{x} \frac{q_{01}(\rho)d\rho}{\sqrt{x^2-\rho^2}} + i\int_{\rho_3}^{x} \frac{\rho p_{01}(\rho)d\rho}{\sqrt{x^2-\rho^2}}\right) + \frac{Cx+iD}{\sqrt{x+\rho_3}} \tag{3.37}$$

Note that the functions $M_{klij}(x, t)$ (3.35), defined in the squares $\rho_{2j-1} \leqslant x, t \leqslant \rho_{2j}$ ($j = 1, 2$) have a removable singularity on the diagonals $t = x$. The values of these functions when $t = x$ are assumed to be equal to the limiting values when $t \to x$ which are determined using l'Hôpital's rule.

The unknown transforms $\bar{p}_1(\beta)$, $\bar{q}_1(\beta)$ (3.11) and $\bar{f}(\beta)$, $\bar{g}(\beta)$ (3.13) are expressed in terms of the real parts $\varphi_j(x) = \operatorname{Re} \omega_j(x)$ and the imaginary parts $\vartheta_j(x) = \operatorname{Im} \omega_j(x)$ of the complex functions of the real variables $\omega_j(x)$ ($j = 1, 2$), which satisfy the system of singular integral equations (3.30) using the formulae

$$\bar{p}_1(\beta) = \int_{\rho_1}^{\rho_2} \frac{x\varphi_1(x)S_{01}(x,\beta)}{\sqrt{x-\rho_1}}dx, \quad \bar{q}_1(\beta) = \int_{\rho_1}^{\rho_2} \frac{\vartheta_1(x)S_{11}(x,\beta)}{\sqrt{x-\rho_1}}dx \tag{3.38}$$

$$\bar{f}(\beta) = \int_{\rho_3}^{\rho_4} \frac{x\varphi_2(x)S_{02}(x,\beta)}{\sqrt{x-\rho_3}}dx, \quad \bar{g}(\beta) = \int_{\rho_3}^{\rho_4} \frac{\vartheta_2(x)S_{12}(x,\beta)}{\sqrt{x-\rho_3}}dx \tag{3.39}$$

From the theory of the identity transformation of the system of equations (3.18), (3.24), it automatically follows in the system of singular integral equations (3.3), in the case of transforms (3.11) and (3.13), that the transforms $\bar{p}_1(\beta)$, $\bar{q}_1(\beta)$ (3.38) and $\bar{f}(\beta)$, $\bar{g}(\beta)$ (3.39) reduce all the equations of the system (3.18), (3.24) and the initial system (3.18)–(3.20), with the possible exception of Eqs (3.20), to identities. These last equations (3.20) are differentiated with respect to ρ during the transformation process and, therefore, need to be checked by substituting the transforms $\bar{f}(\beta)$ and $\bar{g}(\beta)$ (3.39) into them. A check showed that Eqs (3.20) are only satisfied in the case of the additional conditions [2]

$$\int_{\rho_3}^{\rho_4} \frac{x\varphi_2(x)}{\sqrt{x^2-\rho_3^2}}dx = 0, \quad \int_{\rho_3}^{\rho_4} \frac{\vartheta_2(x)}{\sqrt{x^2-\rho_3^2}}dx = 0 \tag{3.40}$$

from which the arbitrary constants C and D, on the right-hand side of the system of singular integral equations (3.30), appearing in the function $\Phi_2(x)$ (3.37), are determined.

In order to satisfy conditions (3.40), we will seek the solution of the system of singular integral equations (3.30) in the form

$$\omega_j(x) = \omega_{j1}(x) + C\omega_{j2}(x) + D\omega_{j3}(x), \quad j = 1, 2 \tag{3.41}$$

where $\omega_{j1}(x)$, $\omega_{j2}(x)$ and $\omega_{j3}(x)$ are particular solutions of the system of singular integral equations (3.30) which, respectively, take account of its right-hand side: (1) $C = D = 0$, (2) $\Phi_1(x) \equiv 0$, $\Phi_2(x) = x/\sqrt{x+\rho_3}$ ($C = 1$, $D = 0$) and (3) $\Phi_1(x) \equiv 0$, $\Phi_2(x) = i/\sqrt{x+\rho_3}$ ($C = 0$, $D = 1$). On substituting the real part $\varphi_2(x) = \varphi_{21}(x) + C\varphi_{22}(x) + D\varphi_{23}(x)$ and the imaginary part $\vartheta_2(x) = \vartheta_{21}(x) + C\vartheta_{22}(x) + D\vartheta_{23}(x)$ of the function $\omega_2(x)$ (3.41) into equalities (3.4), it is easy to write out the closed system of two algebraic equations in the constants C and D from which they are calculated.

4. THE MIXED PROBLEM OF AN ANNULAR PUNCH AND A CIRCULAR CRACK

The mixed problem of an annular punch and a circular crack is considered as a special case of the problem from Section 3 when $\rho_4 < \rho_1$ and $\rho_3 = 0$. At the same time, it is of interest in its own right. Its formulation and solution repeats all the calculations presented in Section 3. Formally, it reduces to the system of singular integral equations (3.30) on the bounded contour

$$(L_1, L_3^0) = (\rho_1 \leqslant \rho \leqslant \rho_2, \ 0 \leqslant \rho \leqslant \rho_4)$$

which is easily written out by assuming the values of ρ_{2j-1} and ρ_{2n-1} when j, $n = 2$ in all the formulae (3.30), (3.34), (3.35) and (3.37) to be equal to zero. In this case, the form of the following functions is simplified

$$S_{02}(y,\beta)=\cos(y\beta)/\sqrt{y},\quad S_{12}(y,\beta)=\sqrt{y}\sin(y\beta),\quad y=x,t$$
$$M_{0022}(x,t)\equiv M_{1122}(x,t)\equiv 0,\quad M_{0122}(x,t)=M_{1022}(x,t)=\sqrt{xt}/[2(x+t)] \tag{4.1}$$

However, the system of singular integral equations (3.3) on the contour (L_1, L_3^0) is inconvenient for investigating the problem being considered due to the fact that mathematical difficulties are encountered in constructing a closed, analytical solution of the characteristic system of singular integral equations (3.30) on the contour (L_1, L_3^0) in the appropriate class of functions (continuous and bounded at the crack centre $\rho = 0$ and unbounded at the end $\rho = \rho_4$) and, consequently, regularization of the initial complete system of singular integral equations by the Karleman–Vekua method. In order to avoid these difficulties. It is best to transform identically the system of singular integral equations (3.30) on the contour (L_1, L_3^0) into an equivalent system of singular integral equations on the contour

$$(L_1,L_3^*)=(\rho_1\leqslant\rho\leqslant\rho_2,\ -\rho_4\leqslant\rho\leqslant\rho_4) \tag{4.2}$$

by writing the first equation on the contour $L_1 = (\rho_1 \leqslant \rho \leqslant \rho_2)$ when $\rho_3 = 0$ and the second equation on the contour $L_3^* = (-\rho_4 \leqslant \rho \leqslant \rho_4)$. In this case, the functions $M_{1022}(x, t)$ and $M_{0122}(x, t)$ (4.1), when account is taken of the evenness of $\varphi_2(t)$ and the oddness of $\vartheta_2(t)$, change from the regular part into the singular part of the system of equations, the continuity of the new kernels.

$$\tilde{H}_{12}^{\pm}=H_{12}^{\pm}(x,t)/\sqrt{t},\quad \tilde{H}_{21}^{\pm}=H_{21}^{\pm}(x,t)/\sqrt{x},\quad \tilde{H}_{22}^{\pm}=H_{22}^{\pm}(x,t)/\sqrt{xt}$$

is taken into account and the free function $\Phi_2(x)$ (3.37) is replaced by the function

$$\tilde{\Phi}_2(x)=-x\int_0^x\frac{q_{01}(\rho)d\rho}{\sqrt{x^2-\rho^2}}+i\int_0^z\frac{\rho p_{01}(\rho)d\rho}{\sqrt{x^2-\rho^2}}+C+i\frac{D}{x} \tag{4.3}$$

On taking account of what has been said above, writing out the system of singular integral equations (3.3) on the contour (L_1, L_3^*) does not present any difficulties whatsoever.

The required transforms $\bar{p}_1(\beta)$, $\bar{q}_1(\beta)$ and $\bar{f}(\beta)$, $\bar{g}(\beta)$ are defined in terms of the real parts $\varphi_j(x)$ and imaginary parts $\vartheta_j(x)$ of the complex functions of the real variable $\omega_j(x)$ ($j = 1, 2$), which satisfy the system of singular integral equations (3.30) on the contour (L_1, L_3^*) (4.2) using formulae (3.38) and

$$\bar{f}(\beta)=\int_0^{\rho_4}\varphi_2(x)\cos(x\beta)dx,\quad \bar{g}(\beta)=\int_0^{\rho_4}\vartheta_2(x)\sin(x\beta)dx \tag{4.4}$$

Substitution of the transforms $\bar{g}(\beta)$, $\bar{f}(\beta)$ (4.4) into Eqs (3.20) shows that the first equation of (3.20) is automatically satisfied while the second equation is only satisfied subject to the additional condition

$$\int_0^{\rho_4}\varphi_2(x)dx=0 \tag{4.5}$$

from which the arbitrary constant C, occurring in the function $\tilde{\Phi}_2(x)$ (4.3), is determined. The second constant D in relation (4.3), as an un-called for constant, is taken as being equal to zero.

In order to satisfy condition (4.5), we will seek the solution of the system of singular integral equations (3.30) on the contour (L_1, L_3^*) (4.2) in the form

$$\omega_j(x)=\omega_{j1}(x)+C\omega_{j2}(x),\quad j=1,2 \tag{4.6}$$

where $\omega_{j1}(x)$, $\omega_{j2}(x)$ ($j = 1, 2$) are particular solutions of the above-mentioned system of singular integral equations taking into account, respectively, (1) $C = 0$, $D = 0$ and (2) $\Phi_1(x) \equiv 0$, $\Phi_2(x) = 1$ ($C = 1$, $D = 0$) on its right-hand side. On substituting the real part $\varphi_2(x) = \varphi_{21}(x) + C\varphi_{22}(x)$ of the function $\omega_2(x)$ (4.6) into equality (4.5), it is easy to write out the linear equation for determining the constant C.

5. THE MIXED PROBLEM OF A CIRCULAR PUNCH AND AN ANNULAR CRACK

The mixed problem of a circular punch and an annular crack is considered as a special case of the problem from Section 3 when $\rho_3 > \rho_2$ and $\rho_1 = 0$. In this case, the form of the following functions are simplified

$$S_{01}(y,\beta) = \cos(y\beta)/\sqrt{y}, \quad S_{11}(y,\beta) = \sqrt{y}\sin(y\beta), \quad y = x,t$$
$$M_{0011}(x,t) \equiv M_{1111}(x,t) \equiv 0, \quad M_{0111}(x,t) \equiv M_{1011}(x,t) \equiv \sqrt{xt}/[2(x+t)] \tag{5.1}$$

By analogy with the problem from Section 4, it is advisable to transform the system of singular integral equations (3.30) to the equivalent system of singular integral equations on the contour

$$(L_1^*, L_3) = (-\rho_2 \leqslant \rho \leqslant \rho_2, \ \rho_3 \leqslant \rho \leqslant \rho_4) \tag{5.2}$$

by writing the first equations on the contour $L_1^* = (-\rho_2 \leqslant \rho \leqslant \rho_2)$ and the second equation on the contour $L_3 = (\rho_3 \leqslant \rho \leqslant \rho_4)$ when $\rho_1 = 0$. At the same time, when account is taken of the evenness of $\varphi_1(t)$ and the oddness of $\vartheta_1(t)$, the functions $M_{0111}(x, t), M_{1011}(x, t)$ (5.1) change from the regular into the singular part of the system of equations, the continuity of the new kernels

$$\tilde{H}_{11}^{\pm} = H_{11}^{\pm}(x,t)/\sqrt{xt}, \quad \tilde{H}_{12}^{\pm} = H_{12}^{\pm}(x,t)/\sqrt{x}, \quad \tilde{H}_{21}^{\pm} = H_{21}^{\pm}(x,t)/\sqrt{t} \tag{5.3}$$

is taken into account and the free function $\Phi_1(x)$ (3.36), taking account of the equality $\gamma(0) = 0$, is replaced by the function

$$\tilde{\Phi}_1(x) = -h + x\int_0^x \frac{\gamma'(\rho)d\rho}{\sqrt{x^2-\rho^2}} \tag{5.4}$$

The required transforms $\bar{p}_1(\beta), \bar{q}_1(\beta)$ and $\bar{f}(\beta), \bar{g}(\beta)$ are defined in terms of the solution of the system of singular integral equations (3.3) on the contour (L_1^*, L_3) (5.2) $\omega_j(x)$ $(j = 1, 2)$ in the form (3.41) using formulae (3.38) and (3.39) when $\rho_1 = 0$ taking account of the functions $S_{01}(y, \beta), S_{11}(y, \beta)$ (5.1).

The arbitrary constants C and D, occurring in the functions $\Phi_2(x)$ (3.37), are determined using the technique employed in Section 3 from conditions (3.40). At the same time, the functions $\Phi_1(x)$ (3.36) is replaced by $\tilde{\Phi}_1(x)$ (5.4).

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